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A simple derivation of the exact wavefunction of a harmonic oscillator with time-dependent mass and frequency

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Abstract. We present a very simple and efficient method to obtain the exact wavefunction corresponding to the harmonic oscillator with time-dependent mass and frequency.

1. Introduction

The study of harmonic oscillators with time-dependent frequencies or with time-dependent masses (or both simultaneously) has attracted considerable interest in the past few years [1–6]. Apart from its intrinsic mathematical interest, the time-dependent harmonic oscillator has invoked much attention because of its many applications in different areas of physics, such as quantum optics and plasma physics. For instance, it has been shown [7] that the Hamiltonian describing the problem of a Fabry–Perot cavity in contact with a heat reservoir assumes the form of a harmonic oscillator with constant frequency and time-dependent mass.

Such a problem arises naturally from the quantum treatment of the damped harmonic oscillator and has been studied by the use of a canonical transformation [8] which transforms the time-dependent mass oscillator to one with a time-dependent frequency. This method produces an exact solution for the time-dependent Schrödinger equation in both the Schrödinger and the Heisenberg picture. The advantage of being able to treat particles with time-dependent masses shows up most obviously in the case of a particle which is decaying and losing mass under the influence of a time-dependent gravitational potential [9] and also in plasma physics [10].

The case of the harmonic oscillator with constant mass and time-dependent frequency has also attracted much attention and its exact wavefunction has been obtained [2] by applying a path-integral method.

Many techniques have been devised to study quantum systems whose Hamiltonians are explicitly time dependent and among them the path-integral method [11] and the quantum invariant operator method [1] of Lewis and Riesenfeld (LR) have been particularly successful. For explicitly time-dependent harmonic oscillators, LR have introduced an important quantum mechanical invariant and found the exact quantum states in terms of the invariant eigenstates.

Recently, much attention has been dedicated to the most general case of a harmonic oscillator with time-dependent mass and frequency [3, 5, 6, 12] which constitutes a rather difficult, but yet one of the few solvable problems in this field. All these treatments are essentially based on the LR invariant method which, although a powerful tool for the study of such systems, is not very easy to apply. Within this method the solution of the nonlinear equation for the parameter entering the LR invariant equation still remains a difficult task.

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Because of such mathematical difficulties associated with the LR method, some results corresponding to the harmonic oscillator with time-dependent mass and frequency, which have appeared in the recent literature, do not agree with each other and it seems that a few of them [3, 12] are not quite correct.

The main purpose of the present paper is to exhibit a new solution of the general problem of the harmonic oscillator with time-dependent mass and frequency by employing some simple transformations of variables. The derivation of the exact wavefunction is straightforward and is obtained with much less effort than other results [6] based on the LR invariant method.

2. The time-dependent harmonic oscillator

The time-dependent Schrödinger equation for the harmonic oscillator with time-dependent mass and frequency is written as

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \Psi(x,t) = \hat{H}(t)\,\Psi(x,t) \tag{1}$$

with the time-dependent Hamiltonian operator given by

$$\hat{H}(t) = -\frac{\hbar^2}{2m(t)}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m(t)\,\omega(t)^2 x^2 \tag{2}$$

where $m(t)$ is the time-dependent mass, $\omega(t)$ is the time-dependent frequency and $\Psi(x,t)$ is the space- and time-dependent wavefunction which is the solution of the time-dependent Schrödinger equation.

The simplest solution of equation (1) is obtained [13] when both mass $m(t) = m_0$ and frequency $\omega(t) = \omega_0$ are constant and is given by

$$\Psi_n(x,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m_0\omega_0}{\hbar\pi}\right)^{1/4} \exp\left[-\mathrm{i}(n+\tfrac{1}{2})\,\omega_0 t\right] \exp\left[-\frac{m_0}{2\hbar}\omega_0 x^2\right] H_n\left(\sqrt{\frac{m_0\omega_0}{\hbar}}x\right) \tag{3}$$

where $H_n(x) = (-1)^n \exp(x^2)(\partial^n/\partial x^n)[\exp(-x^2)]$ are the Hermite polynomials and $n = 0, 1, \ldots$ is a non-negative integer.

To obtain an exact solution of the harmonic oscillator with time-dependent mass and frequency, let us write the function $\Psi(x,t)$ in terms of a new wavefunction $\Phi(x,t)$ as

$$\Psi(x,t) = \exp[-\alpha(t)\,x^2 - \beta(t)]\Phi(x,t) \tag{4}$$

where the time-dependent functions $\alpha(t)$ and $\beta(t)$ are to be found later. By substituting equation (4) into (1) one readily obtains

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial}{\partial t}\Phi(x,t) = {} & -\frac{\hbar^2}{2m(t)}\frac{\partial^2}{\partial x^2}\Phi(x,t) + \left[\frac{1}{2}m(t)\,\omega(t)^2 - \frac{2\hbar^2}{m(t)}\alpha(t)^2 + \mathrm{i}\hbar\dot{\alpha}(t)\right]x^2\Phi(x,t) \\ & + \left[\mathrm{i}\hbar\dot{\beta}(t) + \frac{\hbar^2}{m(t)}\alpha(t)\right]\Phi(x,t) + \frac{2\hbar^2}{m(t)}\alpha(t)\,x\frac{\partial}{\partial x}\Phi(x,t)\end{aligned} \tag{5}$$

where $\dot{\alpha}(t)$ and $\dot{\beta}(t)$ denote the time derivatives over the functions $\alpha(t)$ and $\beta(t)$.

By introducing a new variable y related to x through the relation $x = \rho(t)\,y$ where $\rho(t)$ is a time-dependent function, the function $\Phi(x,t)$ is transformed into a new function $\Phi'(y,t)$ in such a way that

$$\Phi(x,t) = \Phi'\left(y = \frac{x}{\rho(t)}, t\right). \tag{6}$$

Using the chain rule for partial derivatives we find that in terms of the new variable y the following quantities are written as

$$\frac{\partial}{\partial t}\Phi(x,t) = \frac{\partial}{\partial t}\Phi'(y,t) - y\frac{\dot{\rho}(t)}{\rho(t)}\frac{\partial}{\partial y}\Phi'(y,t)$$

and is straightforward to note that

$$x\frac{\partial}{\partial x}\Phi(x,t) = y\frac{\partial}{\partial y}\Phi'(y,t)$$

$$\frac{\partial^2}{\partial x^2}\Phi(x,t) = \frac{1}{\rho(t)^2}\frac{\partial^2}{\partial y^2}\Phi'(y,t).$$

In terms of the new function $\Phi'(y,t)$ and the new variable y we can rewrite equation (5) as

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial t}\Phi'(y,t) = &-\frac{\hbar^2}{2m(t)\rho(t)^2}\frac{\partial^2}{\partial y^2}\Phi'(y,t) \\ &+\left[\frac{1}{2}m(t)\,\omega(t)^2 - \frac{2\hbar^2}{m(t)}\alpha(t)^2 + \mathrm{i}\hbar\dot{\alpha}(t)\right]\rho(t)^2 y^2\Phi'(y,t) \\ &+\left[\mathrm{i}\hbar\dot{\beta}(t) + \frac{\hbar^2}{m(t)}\alpha(t)\right]\Phi'(y,t) + \left[\mathrm{i}\hbar\frac{\dot{\rho}(t)}{\rho(t)} + \frac{2\hbar^2}{m(t)}\alpha(t)\right]y\frac{\partial}{\partial y}\Phi'(y,t). \end{aligned} \tag{7}$$

This equation looks very complicated, but it can be enormously simplified by choosing the auxiliary time-dependent functions $\alpha(t)$, $\beta(t)$ and $\rho(t)$ in such a way that the coefficients appearing in front of $\Phi'(y,t)$ and $y(\partial/\partial y)\Phi'(y,t)$ vanish, and the coefficient in front of $y^2\Phi'(y,t)$ is equal to $1/[2m(t)\rho(t)^2]$.

By combining these results we can easily find that $\rho(t)$ must satisfy the second-order differential equation

$$\ddot{\rho}(t) + \frac{\dot{m}(t)}{m(t)}\dot{\rho}(t) + \omega(t)^2\rho(t) = \frac{1}{m(t)^2\rho(t)^3} \tag{8}$$

where $\alpha(t)$ and $\beta(t)$ are given in terms of $\rho(t)$ as

$$\alpha(t) = -\mathrm{i}\frac{m(t)}{2\hbar}\frac{\dot{\rho}(t)}{\rho(t)} \tag{9}$$

and

$$\dot{\beta}(t) = \mathrm{i}\frac{\hbar}{m(t)}\alpha(t) = \frac{1}{2}\frac{\dot{\rho}(t)}{\rho(t)}. \tag{10}$$

These transformations and the conditions put on the auxiliary time-dependent functions allow us to write equation (7) as

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Phi'(y,t) = -\frac{\hbar^2}{2m(t)\rho(t)^2}\frac{\partial^2}{\partial y^2}\Phi'(y,t) + \frac{y^2}{2m(t)\rho(t)^2}\Phi'(y,t) \tag{11}$$

where $\rho(t)$ is the only auxiliary time-dependent function that appears in equation (11).

A simplification of equation (11) is readily achieved by introducing the new time variable

$$\tau = \int_0^t \frac{\mathrm{d}t'}{m(t')\rho(t')^2} \tag{12}$$

where one transforms equation (11) into a simpler equation of the form

$$\mathrm{i}\hbar\frac{\partial}{\partial \tau}\Phi'(y,\tau) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2}\Phi'(y,\tau) + \frac{1}{2}y^2\Phi'(y,\tau). \tag{13}$$

It is not difficult to recognize that equation (13) corresponds to the harmonic oscillator with constant mass and frequency described from equations (1) and (2) with $m(t) = m_0 = 1$ and $\omega(t) = \omega_0 = 1$ and its exact solution is found in equation (3). As a final step, substituting equations (9) and (10) into equation (4) and returning to the original variables x and t, one obtains an exact solution of equation (1) for the harmonic oscillator with time-dependent mass and frequency in the form

$$\Psi_n(x,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1}{\hbar\pi\rho(t)^2}\right)^{1/4} \exp\left[-\mathrm{i}\left(n+\tfrac{1}{2}\right)\int_0^t \frac{\mathrm{d}t'}{m(t')\rho(t')^2}\right] \times \exp\left[\mathrm{i}\frac{m(t)}{2\hbar}\left(\frac{\dot{\rho}(t)}{\rho(t)} + \frac{\mathrm{i}}{m(t)\rho(t)^2}\right)x^2\right] H_n\left(\sqrt{\frac{1}{\hbar}}\frac{x}{\rho(t)}\right) \tag{14}$$

where the time-dependent function $\rho(t)$ should satisfy equation (8).

For the general case of the harmonic oscillator with time-dependent mass and frequency, the exact Schrödinger wavefunction given in equation (14) agrees with that of Pedrosa [6] obtained using the LR invariant method. It also agrees with that of Ji *et al* [5] obtained using the Heisenberg picture approach and the LR invariant method by simply setting $\rho(t)^2 = g(t)/\omega_I$. As pointed out recently [6], our final result is different from that of Dantas *et al* [3] which is not correct for the general case under consideration. It also differs from that of Kim [12] obtained using the LR invariant method for a specific form of the time-dependent mass and frequency and that misses some time-dependent phase factor. Note that when both the mass and frequency are constant, $m(t) = m_0$ and $\omega(t) = \omega_0$ then $\rho(t) = \rho_0$ where $\rho_0 = 1/\sqrt{m_0\omega_0}$ is a particular solution of equation (8). As a result the solution given in equation (14) becomes the solution of the Schrödinger equation for the harmonic oscillator with constant mass and frequency already given in equation (3).

Since each $\Psi_n(x,t)$ satisfies the time-dependent Schrödinger equation, the general solution of equation (1) may be written as

$$\Psi(x,t) = \sum_{n=0}^{\infty} C_n \Psi_n(x,t) \tag{15}$$

where the C_n are constants. If the initial state of the system at $t = 0$ is one of the stationary states of the harmonic oscillator, then equation (8) should be solved with the initial conditions $\rho(t=0) = 1/\sqrt{m(t=0)\,\omega(t=0)}$ and $\dot{\rho}(t=0) = 0$ that correspond to the correct wavefunction $\Psi_n(x, t=0)$ in equation (14). The $\Psi_n(x, t=0)$ wavefunction is the eigenstate of the instantaneous Hamiltonian $\hat{H}(t=0)$ where one can write

$$\hat{H}(t=0)\,\Psi_n(x,t=0) = E_n(t=0)\,\Psi_n(x,t=0) \tag{16}$$

where $E_n(t=0) = \hbar\omega(t=0)(n+\frac{1}{2})$ is the instantaneous energy eigenvalue. The time evolution of the initial state $\Psi_n(x, t=0)$ is found by solving the time-dependent Schrödinger equation and the general solution is given by equation (15). Since the Hamiltonian is time dependent, these solutions do not enjoy the same privileged status as having constant energy eigenvalues as in the time-independent Hamiltonian case.

3. Summary

The explicitly time-dependent quantum systems have been a long-standing mathematical problem not yet completely solved in general. A good example of such a system that has many applications in different areas of physics is the harmonic oscillator with time-dependent mass and frequency (the most general case). The most preferred tool for studying these systems is the generalized LR invariant method [1], which is based on the idea of constructing quantum invariants and then to find the exact solution of the time-dependent Schrödinger equation in terms of the invariant eigenstates. Since then numerous variants and applications of the LR invariant method have been introduced and used. After lengthy calculations, the exact solution for the harmonic oscillator with time-dependent mass and frequency was obtained recently [5, 6] both in the Schrödinger and Heisenberg representation. These derivations were based on the LR invariant method, which is not easy to apply and mathematically is rather challenging.

In this paper we apply a more intuitive approach to solve the time-dependent Schrödinger equation for the harmonic oscillator with time-dependent mass and frequency by using only simple transformations of variables. Our exact solution is in agreement with that of Pedrosa [6], but was obtained quite differently and with much less effort. We feel that the present paper may stimulate other efforts to search for simpler treatments and solutions of similar problems which until now have been treated only by complicated methods.

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